On a Modification of the Collatz Problem
Projects for the C^2R Research Group

July 28, 2016

The Research Group will investigate iterations of the map $T : \mathbb{N} \to \mathbb{N}$ defined by

$$T(x) = \begin{cases} \dfrac{x}{p}, & \text{if } x \equiv 0 \bmod d, \\ \dfrac{(p+1)x - r(x)}{p}, & \text{if } x \not\equiv 0 \bmod d, \end{cases} \tag{1}$$

where $p > 2$ is a prime number, and $0 < r(x) < p$ is chosen such that $(p+1)x - r(x)$ is divisible by p.

This map was suggested by David Kay in 1972[2]. The literature is rich in other generalizations, see for example [1, 4, 6, 5]. A similar map was studied by M. Paz and M. Ramirez [7]. In particular, the group will work on the following two Projects:

Project 1:

Adapt the results of Paz and Ramirez to this new situation and use these results for an extensive numerical search of "exotic" limit cycles, i.e. limit cycles that are not generated by iterations of $T(1)$. Paz and Ramirez identified a specific subset of the integers, which can give rise to those cycles.

Project 2:

The map T gives rise to a derived map

$$\hat{T} : \mathbb{Z}_p^p \to \mathbb{Z}_p^p.$$

by considering the map modulo p.

$$T_j(x \bmod p) = T(x) \bmod p.$$

The component j depends here on the size of x. This map can be explicitely written down. Studying the properties of this map will give insight to properties of the original map. In particular, we will prove that this map is a bijection and thus reduced to a permutation on a set with p^2 elements. This process was done for the case of the Collatz map ($p = 2$) and can be found in [3].

The first project requires only elementary algebra skills and some programming skills. For the second project the group also needs to be familiar with the basic arithmetic properties of the finite field $\mathbb{Z}_p$. These will be covered in the second week of August 2016.

1 References

References

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[5] Matthews, K. R., Generalized $3x+1$ Mappings: Markov Chains and Ergodic Theory. In The Ultimate Challenge: The $3x+1$ Problem (ed. J.F. Lagarias), American Mathematical Society, 2011, pp. 79-103.

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